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PREPERFECT GRAPHS

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We say that a vertex x of a graph is predominant if there exists another vertex y of G such that either every maximum clique of G containing y contains x or every maximum stable set containing x contains y. A graph is then called preperfect if every induced subgraph has a predominant vertex. We show that preperfect graphs are perfect, and that several well-known classes of perfect graphs are preperfect. We also derive a new characterization of perfect graphs.

Predomination

Our graphs are finite, without loops or multiple edges, and we generally follow the standard terminology of Berge ([2]). The set of all neighbors of a vertex x is denoted by $N(x)$. The size of a maximum clique of G is denoted by $\omega(G)$ and the size of a maximum stable set by $\alpha(G)$. (As usual, "maximal" refers to set-inclusion and "maximum" refers to size.)

Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. If x and y are two vertices of G such that $N(y) \subseteq N(x) \cup \{x\}$, one usually says that x *dominates* y in G, and also that x is a *dominant* vertex of G. It is easily seen that x dominates y if and only if either every maximal clique of G containing y also contains x (if x and y are adjacent) or every maximal stable set of G containing x also contains y (if x and y are not adjacent). Also note that x dominates y in G if and only if y dominates x in the complementary graph. The domination relation between vertices of a graph is a natural and important concept of graph theory, and has been studied in particular by Dilworth ([6]).

We now introduce a generalized form of domination. Let us say that a vertex x of a graph G *predominates* a vertex y if one of the three following situations occurs:

- $V(G)=\{x\}=\{y\}$;
- $x \neq y$, x and y are adjacent, and every *maximum* clique containing y also contains x;

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- $x \neq y$, x and y are not adjacent, and every maximum stable set containing x contains y.

In either case we will say that x is *predominant* and that y is *predominated* in G. Finally we say that a graph G is *predomination-perfect*, or, in short, *preperfect*, if every induced subgraph of G has a predominant vertex.

Berge ([1]) suggested to call a graph G *perfect* if, for every induced subgraph H of G, one can partition the vertex-set $V(H)$ in $\omega(H)$ stable sets. Since the introduction of this concept, perfect graphs have become an important topic of graph theory. The reader will find in [3] a recent collection of important articles on perfect graphs. Berge ([1]) conjectured that a graph is perfect if and only if it does not contain as an induced subgraph an odd chordless cycle with at least five vertices or the complement of such a cycle. This conjecture is still open. A corollary of this conjecture is that the complement of a perfect graph must be perfect, a statement also conjectured by Berge and proved by Lovász ([12], [13]). A formulation of Lovász's result is given now. (A graph is *minimal imperfect* if it is not perfect and all its proper induced subgraphs are perfect.)

Theorem 1.1 (Lovász [13]). *A minimal imperfect graph G has exactly $\alpha(G)\omega(G)+1$ vertices. For every vertex x of G, $G-x$ can be partitioned into $\alpha(G)$ cliques of size $\omega(G)$, as well as into $\omega(G)$ stable sets of size $\alpha(G)$.* ∎

Theorem 1.2. *A minimal imperfect graph does not have a pair x,y of vertices such that x predominates y. Any preperfect graph is perfect.*

Proof of Theorem 1.2. Suppose that some preperfect graph G is not perfect. Without loss of generality we may assume that G is minimal imperfect. Let x and y be any two vertices of G. By Lovász's Theorem, there exists a partition of $G-x$ in $\alpha(G)$ cliques of size $\omega(G)$. One of these cliques contains y and not x. Similarly, there exists a partition of $G-y$ in $\omega(G)$ stable sets of size $\alpha(G)$. One of them contains x and not y. It follows that x does not predominate y. Since x and y can be chosen arbitrarily, we have established that G contains no predominant vertex, a contradiction. ∎

For further reference we remark that the definition of preperfect graphs implies the following properties, which are not difficult to check.

(R1) Any induced subgraph of a preperfect graph is preperfect.

(R2) The complement of any preperfect graph is preperfect.

(R3) The disjoint union of several preperfect graphs is preperfect.

(R4) If G and H are disjoint preperfect graph, then the graph obtained by removing x and adding an edge between every vertex of H and every vertex of $N_G(x)$ is preperfect.

In Section 2 we show that some well-known classes of perfect graphs (i-triangulated graphs and parity graphs) are preperfect. In Section 3 we present an infinite family of graphs that are minimally non-preperfect and yet perfect. In Section 4 we show that the notion of predomination yields a new characterization of all perfect graphs.

Some Classes of Preperfect Graphs

It is natural to wonder which among the well-known classes of perfect graphs are included in the class of preperfect graphs. We first consider the case of bipartite graphs. Note that in an even chordless cycle with at least 6 vertices there exists no dominant vertex; but every vertex is predominant since there is only one maximum stable set containing a given vertex. More generally we have the following theorem. (A vertex with exactly one neighbor is called *pendant*. Note that the neighbor of a pendant vertex is predominant.)

Theorem 2.1. *In a connected bipartite graph, every vertex is either pendant or predominant. Every bipartite graph is preperfect.*

The proof of Theorem 2.1 will use the following result of A. Hajnal.

Theorem 2.2 (Hajnal [9]). *Consider any family of stable sets of size $\alpha(G)$ in a graph G. Let u be the size of their union and i the size of their intersection. Then $u+i \geq 2\alpha(G)$.* ▮

Proof of Theorem 2.1. Let G be a connected bipartite graph with n vertices, and x be any vertex of G. Consider the family of all maximum stable sets of G containing x. Let u be the size of the union of all these stable sets, and i be the size of their intersection. By Hajnal's theorem, we have $u+i \geq 2\alpha(G)$. Since G is bipartite, we know that $\alpha(G) \geq n/2$, whence $u+i \geq n$. We distinguish two cases.

If $i=1$, the preceding inequality gives $u \geq n-1$, which means that there are at least $n-1$ vertices (including x) that lie in a common stable set with x. Hence x has at most one neighbor. Since G must be connected we obtain that x is pendant, and thus its neighbor is a predominant vertex of G.

If $i \geq 2$, there exists a vertex y ($y \neq x$) that lies in all maximum stable sets containing x. Thus x predominates y.

In either case, we have actually shown that every vertex of G is either predominated or predominant. The second sentence of the theorem follows from the first one by induction on the size of G, together with remark (R3) in case G is not connected. ▮

We will now generalize Theorem 2.1 to two important classes of perfect graphs.

A graph G is a *parity* graph ([15], [5]) if every odd cycle of G of length at least five has two crossing chords. A graph is *i-triangulated* ([7]) if every odd cycle of length at least five has two non-crossing chords. These two classes generalize bipartite graphs, and i-triangulated graphs also generalize triangulated graphs. We will show that these two classes are preperfect.

A *block* of a graph is a maximal 2-connected induced subgraph. Two blocks have at most one vertex in common. A *terminal* block is a block that contains at most one cut-vertex of the graph. It is easy to see that the blocks of a graph are connected in a tree-like fashion; thus each graph has at least one terminal block.

Theorem 2.3. *Every i-triangulated graph is preperfect.*

Proof of Theorem 2.3. Let G be an i-triangulated graph. Let H be a maximal connected bipartite induced subgraph of G, and B be a terminal block of H. We distinguish between two cases.

Case 1. *The block B consists of a clique of size 2.*

Let a and b be the vertices of B, in such a way that a is the vertex of B which may have neighbors in $H-B$. In other words, b is a pendant vertex of H. We claim that a dominates b in G. Suppose that this is false. Then there exists a neighbor x of b that is not a neighbor of a. Note that $V(H)\cup\{x\}$ induces a connected induced subgraph of G. By the choice of H, this subgraph is not bipartite. However, H is a bipartite graph, so its vertices can be colored with two colors, say pink and grey, so that no two vertices of the same color are adjacent. Without loss of generality, a is pink and b is grey.

Since $H\cup\{x\}$ is not bipartite x must have a pink neighbor c in $V(H)$. Note that $c\neq a$. By the connectedness of H, there exists a chordless path $P=(a=u_1,\ldots,u_p=c)$ from a to c. We choose c so that the length p of P is as small as possible. Clearly, p is odd and at least 3. Now G contains an odd cycle $Z=(x,b,a=u_1,\ldots,u_p=c,x)$ of length at least five. Since G is i-triangulated, this cycle must have two non-crossing chords. Since P is chordless and b is pendant in H, all the chords of Z must be incident to x. Let $I=\{i\,|\,u_i$ is adjacent to $x\}$. By the choice of c, all the elements of I are even. Letting j and k be the largest two elements of I (with $j<k$), we obtain an odd cycle $(x,u_j,\ldots,u_p,x)$ of length at least five and having only one chord, a contradiction. Thus x cannot exist and the claim that a dominates b in G is established.

Case 2. *The block B is not a clique.*

Let a be the vertex of B which may have neighbors in $V(H)-B$. In B, a is not a pendant vertex, since B is not a clique (of size 2). Thus by Theorem 2.1 there exists a vertex w of $B-a$ such that every maximum stable set of B containing a also contains w. We will show that:

$$N_G(B)\subseteq N_G(a). \tag{1}$$

Let us suppose for now that (1) holds, and consider a maximum stable set S of G containing a. Thus S contains no element of $N_G(a)$. It follows from (1) that S contains no element of $N_G(B)$. However, B is a connected component of $G–N_G(B)$. Consequently, $S\cap B$ must be a maximum stable set of B, and thus $S\cap B$ contains w. So we obtain that any maximum stable set S of G containing a contains w, i.e., a predominates w in G.

We now prove (1). If (1) does not hold, there exists a vertex x of $G-H$ that has a neighbor b in B and such that x is not adjacent to a. As before, we may assume that the vertices of H are colored pink and grey, and that b is grey. Since $V(H)\cup\{x\}$ induces a connected subgraph, by the choice of H, this subgraph is not bipartite. Thus x is adjacent to some pink vertex c of H. If x has no pink neighbor in B, then we choose b (in B) and c (in $H-B$) so that b and c are linked by as short as possible a path. This situation is similar to that of Case 1, and leads to a contradiction. So we can assume that $c\in B$.

Since B is 2-connected, by Menger's Theorem there must exist two internally vertex-disjoint paths in B linking respectively b and c to a. Because of the properties of a, b and c, we can choose vertices a^*, b^*, and c^* in B such that: b^* is grey; c^* is pink; x is adjacent to b^* and c^* and not to a^*; b^* and c^* are linked to a^* by two internally disjoint paths P and Q; the total length of P and Q is minimal (thus P and Q are chordless).

Let $P=(b^*=u_0,\dots,u_p=a^*)$ and $Q=(c^*=v_0,\dots,v_q=a^*)$. Note that $p+q$ is odd. As a consequence, the cycle $Z=(x,u_0,\dots,u_p=v_q,\dots,v_0,x)$ is odd and has at least five vertices. Since G is i-triangulated, Z must have at least two non-crossing chords. We examine those chords.

If Z has a chord u_iv_j with $i+j\neq 0$, note that i and j must have the same parity since B is bipartite. Thus one of u_i and v_j is grey and the other one is pink. If x is adjacent to both u_i and v_j, then $\{a^*,u_i,v_j\}$ contradict the choice of $\{a^*,b^*,c^*\}$. If x is not adjacent to u_i or to v_j then either $\{u_i,b^*,c^*\}$ or $\{v_j,b^*,c^*\}$ contradict the choice of $\{a^*,b^*,c^*\}$. So the only possible u_iv_j-chord of Z is $u_0v_0=b^*c^*$, and all other chords of Z must be incident to x.

If x is adjacent to some vertex v_j with j even and $j\geq 2$, then v_j is pink and thus $\{a^*,b^*,v_j\}$ contradict the choice of $\{a^*,b^*,c^*\}$. If x is adjacent to some vertex v_j with j odd and $j\geq 3$, then $\{v_{j-1},v_j,c^*\}$ contradict the choice of $\{a^*,b^*,c^*\}$. It follows that the only possible neighbor of x in $\{v_1,\dots,v_q\}$ is v_1. Similarly, we can show that the only possible neighbor of x among $\{u_1,\dots,u_p\}$ is u_1. Now for Z to have two non-crossing chords the only possibility is that the chords xu_1 and xv_1 both exist. But then $\{a^*,v_1,u_1\}$ contradict the choice of $\{a^*,b^*,c^*\}$. Consequently, the existence of a vertex like x is impossible, and (1) is proved. ∎

Theorem 2.4. *Every parity graph is preperfect.*

For the proof of Theorem 2.4, we will use the construction of parity graphs due to Burlet and Uhry ([5]). They proved that a parity graph can be constructed from a single vertex by repeated use of the following procedures:

Duplication of a vertex. Let G be a parity graph, and x be a vertex of G. Create a new vertex x' and add all edges between x' and all vertices of the set $N_G(x)\cup\{x\}$. Vertex x' is called a *duplicate* (or *true twin*) of x in the new graph.

Extension by a bipartite graph. Let G_1 be a parity graph and x be a vertex of G_1. Let H be a bipartite graph with bipartition (A,B), such that $V(H)\geq 2$. Remove x, choose a subset X of A, and add all possible edges between $N_{G_1}(x)$ and X.

Proof of Theorem 2.4. Let G be a parity graph. Following Burlet and Uhry's construction, we can break up the proof in two cases.

In the first case, G has a pair of duplicate vertices x and x'. Since the vertices x and x' are adjacent, it is easy to see that every maximal clique containing x contains x', and vice-versa. Thus x (pre)dominates x' and vice-versa.

In the second case, G is the result of the extension of a parity graph G_1 by a bipartite graph H. Let (A,B) be a bipartition of H, let X be a subset of A, and x be a vertex of G_1 as in the above definition of the extension by a bipartite. If G_1 is a bipartite graph, with bipartition (A_1,B_1) and $x\in A_1$, then G is a bipartite graph with bipartition $(A\cup A_1-\{x\},B\cup B_1)$. Thus G is preperfect by Theorem 2.1.

So we may assume that G_1 is not bipartite. It follows that in G_1, and consequently in G, every maximum clique has size at least 3. If $A\cup B-X$ is not empty, let y be any vertex of this set. Since $A\cup B-X$ induces a bipartite subgraph of H, and since there is no edge between this subgraph and the rest of G, we conclude that y does not belong to any clique of size 3. It follows that y is predominated in G (by any other vertex of G). On the other hand, if $A\cup B-X$

is empty, then X is a set of at least two pairwise non-adjacent vertices which have exactly the same neighborhood ($=N_{G_1}(x)$). Consequently, any maximal stable set containing a vertex of X must include X, i.e., the vertices of X dominate each other.

In either case, we have seen that G contains a predominant vertex. Thus the proof is complete. ∎

Meyniel ([14]) defined an *even pair* of a graph G to be a pair of vertices $\{x,y\}$ such that every chordless (x,y)-path of G has even length. The graphs in which every induced subgraph or its complement has an even pair are called *quasi-parity*. Meyniel proved that every quasi-parity graphs is perfect. Several well-known classes of perfect graphs are included in the class of quasi-parity graphs (see in particular [10] and [11]). We formulate the following conjecture.

Conjecture. *Every quasi-parity graph is preperfect.*

Consider the graph obtained by taking the Cartesian product of two cliques of size 3 and removing one vertex. It is easy to check that this graph is preperfect, and that it is not a quasi-parity graph. Thus the above conjecture suggests a strict inclusion.

Non-Preperfect Graphs

Since all preperfect graphs are perfect, it follows that a preperfect graph cannot contain an odd chordless cycle of length five (usually referred to as *odd hole*), or the complement of an odd hole (= an odd *anti-hole*). If we delete a vertex from an odd hole or anti-hole, we obtain a subgraph which is either bipartite or the complement of a bipartite graph. By Remark (R2) and Theorem 2.1, this subgraph is preperfect. Thus odd holes and odd anti-holes are minimal non-preperfect graphs (we know that they are minimal imperfect). We present here an infinite family of minimal non-preperfect graphs that are perfect.

Let C_n be the chordless cycle on n vertices $x_1,\dots,x_n$. Let C^*_{4k+2} be the graph obtained from C_{4k+2} by adding an edge between any two opposite vertices on the cycle (i.e., the vertices i and $i+2k+1$ for each $i=1,\dots,2k+1$). Thus C^*_{4k+2} has $6k+3$ edges. Let H_{6k+3} be the line-graph of C^*_{4k+2}, i.e., every vertex of H_{6k+3} represents an edge of C^*_{4k+2}, and two vertices are adjacent if they represent incident edges of C^*_{4k+2}. We claim that H_{6k+3} is a minimal non-preperfect graph, for all $k\geq 1$. First notice that any maximum clique of H_{6k+3} consists of the three vertices representing the edges incident with a given vertex of C^*_{4k+2}. Moreover, any vertex of H_{6k+3} belongs to exactly two such cliques, and these two cliques have no other common vertex. Second, note that a maximum stable set of H_{6k+3} corresponds to a maximum matching of C^*_{4k+2}. It is tedious but straightforward to check that, for any two edges e and f of C^*_{4k+2}, there exists a maximum matching containing e and not f. Consequently, no vertex of H_{6k+3} is predominant. Finally, when some vertices are removed from H_{6k+3}, the remaining subgraph either is bipartite or else has at least one vertex belonging to only one clique of size 3, and thus in either case this subgraph is preperfect.

It is natural to wonder whether there exists any minimal non-preperfect graph besides the above mentioned families (odd holes and anti-holes, graphs of the type H_{6k+3} and their complements). This question remains open.

A related question is the recognition of preperfect graphs. We would like to have an algorithm that determines whether a given graph is preperfect or not in time polynomial in the size of the input. Unfortunately such an algorithm seems difficult to devise, unless some appropriate property of preperfect graphs is found, because the definition of preperfection involves the stability number and clique number of the graph and of each of its subgraphs. We do not even know how to find in polynomial time one predominant vertex in a graph. (Incidentally, we do not know how to recognize in polynomial time the graphs of which every induced subgraph possesses a dominant vertex.)

However, we claim that the problem of recognizing preperfect graphs is in co-NP. Indeed, it suffices to produce, for each non-preperfect graph G, a "certificate of non-preperfectness" that can be verified in polynomial time. Such a certificate has the following two-tier format:

- If G is imperfect, we exhibit a subgraph H of G with the following property: there exist two integers $p,q \geq 2$, a family of $pq+1$ stables sets of H of size p and a family of $pq+1$ cliques of H of size q, such that H has exactly $pq+1$ vertices and each vertex belongs to exactly p stable sets and to exactly q cliques of the family. The fact that H is imperfect derives from the results of Lovász and Padberg ([12], [16]; see also [8]).
- If G is perfect, we exhibit an induced subgraph H of G, together with:
  - (i) A partition of H into ω stable sets;
  - (ii) A partition of H into α cliques;
  - (iii) For each pair of vertices x,y of H, a stable set S_x of H containing x and not y, and a stable set S_y of H containing y and not x, with $|S_x|=|S_y|=\alpha$;
  - (iv) For each pair of vertices x,y of H, a clique K_x of H containing x and not y, and a clique K_y of H containing y and not x, with $|K_x|=|K_y|=\omega$.

Clearly these data can be checked in polynomial time and imply that G is not preperfect.

Absorbant Sets in Perfect Graphs

In a graph $G=(V,E)$, a set of vertices A is called *absorbant* if every vertex of $V-A$ has at least one neighbor in A. Trivially, V is an absorbant set and every set containing an absorbant set is itself absorbant. Thus it is more interesting to consider the notion of *minimal absorbant* set, i.e., an absorbant set A such that every proper subset of A is not absorbant. Notice that every maximal stable set of G is minimal absorbant. We recall that a stable set S of a graph G is said to be *strong* if it has a non-empty intersection with every maximal clique of G. A graph G is then called *strongly perfect* if every induced subgraph of G possesses a strong stable set. Berge and Duchet ([4]) introduced strongly perfect graphs as a special subclass of perfect graphs, and they showed that several classical families of perfect graphs, such as triangulated graphs and transitively orientable graphs, are actually strongly perfect. We define here a generalization of this class.

We will say that a graph G is *absorbantly perfect* if every induced subgraph H of G contains a minimal absorbant set that meets all maximal cliques of H. It follows from the observation above that every strongly perfect graph is absorbantly perfect. While the perfectness of strongly perfect graphs follows directly from their definition, the fact that absorbantly perfect graphs too are perfect, which we prove now, is less obvious.

Theorem 4.1. *Every absorbantly perfect graph is perfect.* ∎

We will not give a direct proof of Theorem 4.1; instead, we will prove the following lemma which is slightly stronger. It is easy to see that Lemma 4.2, together with the fact that a minimal imperfect graph cannot contain a dominant vertex, implies Theorem 4.1.

Lemma 4.2. *Every absorbantly perfect graph possesses either a strong stable set or a dominant vertex.*

Proof of Lemma 4.2. Let G be an absorbantly perfect graph. We will show that the statement of the lemma holds true for G by induction on the number $|V|$ of vertices of G. The fact is trivial when $|V|$ is small.

Let A be a minimal absorbant set of vertices of G that meets every maximal clique of G. If A is a stable set, then A is a strong stable set of G, and we are done. So we may assume that A is not stable, and there exists an edge ab of G with $a,b \in A$.

Since A is a minimal absorbant set, the set $A-a$ is not absorbant, and so there must be a vertex $x \in V-(A-a)$ having no neighbor in $A-a$. Clearly $x \neq a$ since a has a neighbor b in $A-a$. Thus $x \in V-A$. Since A is absorbant in G, vertex x must have a neighbor in A. It follows that a is the unique neighbor of x in A. Now we will show that a dominates x in G. Consider any maximal clique C of G containing x. By the definition of A, this clique must have a non-empty intersection with A. However, any vertex c of $C \cap A$ is adjacent to x and thus $c=a$. So we have proved that every maximal clique C containing x must also contain a, i.e., that a dominates x. ∎

The graphs shown on Figure 1 are absorbantly perfect and not strongly perfect. Therefore, the class of absorbantly perfect graphs is strictly larger than the class of strongly perfect graphs.

The problem of the recognition of strongly perfect graphs is a difficult one and is still unsolved. Similarly, all we can say here is that the complexity status of the recognizing absorbantly perfect graphs remains an open question.

We now turn to the idea of intersecting all maximum cliques of G with a minimal absorbant set. We will see that this idea yields an alternate characterization of perfect graphs.

Theorem 4.3. *A graph G is perfect if and only if every induced subgraph H of G possesses a minimal absorbant set that meets all maximum cliques of H.*

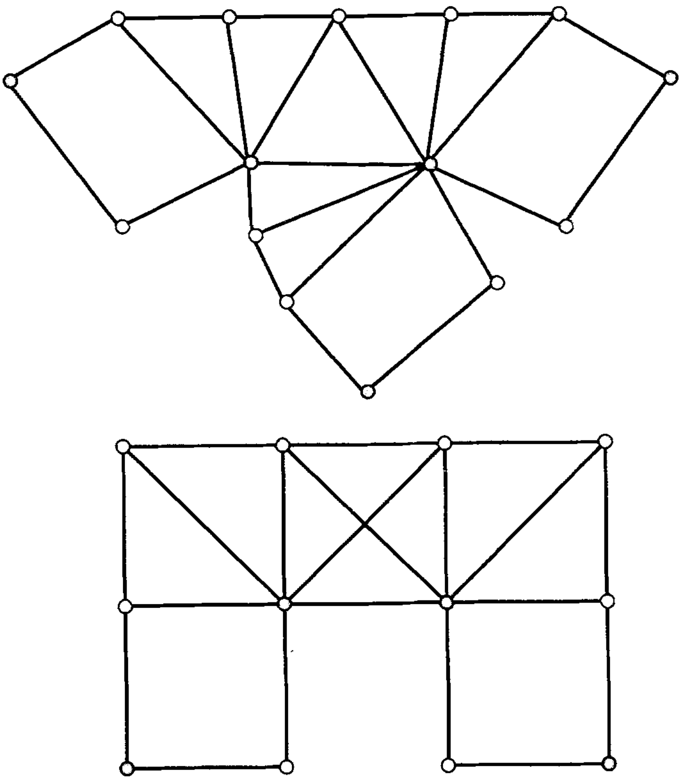

Fig. 1. Grey vertices form a minimal absorbant set that meets all maximal cliques of G.

Proof of Theorem 4.3. The only if part is obvious: it follows from the definition that every perfect graph has a maximal stable set (hence a minimal absorbant set) that meets all maximum cliques.

For the proof of the if part, let us consider a graph G with the property that every induced subgraph H of G possesses a minimal absorbant set that meets all maximum cliques of H, and assume without loss of generality that G is minimal imperfect.

Consider a minimal absorbant set A of vertices that meets all maximum cliques of G. Such a set exists by the assumption. If A is a stable set we are done because a minimal imperfect graph cannot contain a stable set that meets all maximum cliques of G. So we may assume that A is not stable, i.e., there exists an edge ab of G with $a, b \in A$. By the definition of A, the set $A-a$ is not absorbant. Therefore, just like in the proof of Theorem 4.1, there exists a vertex x in $V-A$ with the property that a is the only neighbor of x in A. We now show that a predominates x. For this purpose, we consider a maximum clique C of G containing x. Since A meets every maximum clique of G, there is an element in the set $A \cap C$. Since this element must be a neighbor of x, it can only be a. Thus we have obtained that every maximum clique containing x contains a, and so a predominates x. It follows from Theorem 1.2 that G cannot be minimal imperfect, a contradiction. ∎

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